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BIFURCATIONS OF THE PHASE PATTERN OF AN EQUATION SYSTEM ARISING IN THE PROBLEM OF STABILITY LOSS OF SELFOSCILLATIONS CLOSE TO 1:4 RESONANCE

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A parameter-dependent system of differential equations on a plane, arising in the problem of loss of stability of a periodic solution close to 1:4 resonance, is analyzed. Its phase pattern types are described in the cases when this system is almost– Hamiltonian.

1. Equations in polar coordinates. It was shown in [1] that it is necessary to study the bifurcations of the phase pattern of the equation

$$z^{\cdot} = \varepsilon z + Az\,|\,z\,|^{2} + B\bar{z}^{3} \tag{1.1}$$

where $z = x + iy$ is a point in the complex plane and ε, A, B are complex parameters, in order to describe the phenomena arising from the loss of stability of the periodic solution. In the present paper we describe the phase pattern types of (1.1), arising for small $\operatorname{Re}\varepsilon$ and $\operatorname{Re}A$. The main questions here are connected with ascertaining dispositions and the bifurcations of the limit cycles.

Let us describe briefly the connection between the stability loss problem and Eq. (1.1). Suppose that in a parameter-dependent differential equation system there is, for certain parameter values, a periodic solution all of whose multipliers lie in the unit disk. Suppose that under a variation of the parameters the solution being examined loses stability in the following manner; a pair of complex-conjugate multipliers intersect the unit circle, while the rest lie in the unit disk. At the instant of stability loss and at instants close to this bifurcations necessarily take place in a neighborhood of the periodic solution being examined: other periodic solutions and two-dimensional invariant tori are generated and disappear. When the multipliers intersect the unit circle not too close to the points $\pm i$, these bifurcations have been described in [1, 2]. The case of multipliers close to $\pm i$ has not yet been studied fully. The motion in a neighborhood of a periodic solution is studied by analyzing the normal form of the differential equation system around this solution. If the multipliers are close to $\pm i$, then when constructing the normal form we need to take into account the 1:4 resonance between the motion with respect to the original periodic solution and the oscillations of the solutions of the variational equations around it. Then in the main approximation Eq. (1.1) splits off into normal form. The parameter ε in it describes the deviation of the multiplier from point i. A periodic solution corresponds to the equilibrium $z = 0$, periodic solutions close to the original one but with a period approximately four times larger correspond to the other equilibria, and two-dimensional invariant tori of the original equation system correspond to the limit cycles of (1.1). The bifurcations in a neighborhood of the periodic solution can be described by studying the phase pattern of (1.1).

The parameter B in (1.1) can be made real by rotating the phase plane (x, y). We denote $\varepsilon = \sigma + i\tau$, $A = -\gamma - i\alpha$, $B = \beta$. Following [1], we introduce symplectic polar coordinates ρ and φ and we rewrite the original Eq. (1.1) as the system

$$\rho^{\cdot} = -\partial H/\partial\varphi + 2\rho(\sigma - 2\gamma\rho), \quad \varphi^{\cdot} = \partial H/\partial\rho \qquad (1.2)$$
$$(\rho = |z|^2/2, \quad \varphi = \arg z, \quad H = \tau\rho - \rho^2(\alpha + \beta\sin 4\varphi))$$

This system is invariant relative to a rotation of the phase plane through an angle $\pi/2$. We assume that $\tau \neq 0$, $\beta \neq 0$. Then, τ and β can be made positive by reversing the time direction and rotating the phase plane through an angle $\pi/4$. Therefore, we take it that $\tau > 0$, $\beta > 0$.

2. **Phase pattern of the unperturbed problem** ($\sigma = \gamma = 0$). In what follows we examine the case of small σ and γ. Therefore, at first we describe the problem's phase plane for $\sigma = \gamma = 0$. For such σ and γ system (1.2) takes the Hamiltonian form

$$\rho^{\cdot} = -\partial H/\partial\varphi, \quad \varphi^{\cdot} = \partial H/\partial\rho \qquad (2.1)$$

Using the integral $H = \text{const}$ it can be shown that the phase pattern of system (2.1) can be of the following three forms, depending upon the relation between parameters α and β.

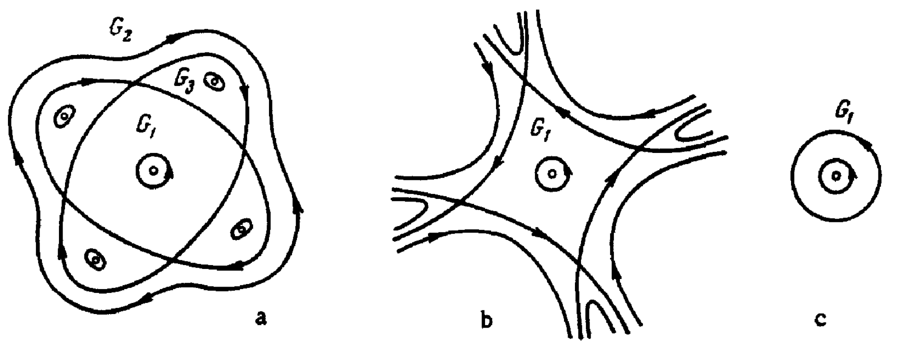

Fig. 1

A) $\alpha > \beta$. The phase pattern is shown in Fig. 1a. The four saddles have the coordinates $\rho = \rho_c = {}^1/_2\tau/(\alpha+\beta)$, $\varphi = \pi/8 + \pi n/2$ $(n = 1, \ldots, 4)$; at the saddles $H = h_c = {}^1/_4\tau^2/(\alpha+\beta)$. The four centers have the coordinates $\rho = \rho_a = {}^1/_2\tau/(\alpha-\beta)$, $\varphi = 3\pi/8 + \pi n/2$ $(n = 1, \ldots, 4)$; at these centers $H = h_a = {}^1/_4\tau^2/(\alpha-\beta)$. The origin too is a center. The separatrices form two like ellipses with mutually perpendicular major semiaxes. The separatrices divide the plane into the regions G_1, G_2, $G_3^{(n)}$ $(n = 1,\ldots, 4)$, filled with closed trajectories. Any of the regions $G_3^{(n)}$ will be referred to as region G_3. The trajectory equations are found from the relation $H = h = \text{const}$, where $0 \leqslant h < h_c$ in region G_1, $-\infty < h < h_c$ in region G_2, and $h_c < h \leqslant h_a$ in region G_3. In each of the regions the trajectory is uniquely determined by the value of h.

B) $\beta \geqslant \alpha > -\beta$ (Fig. 1b). The saddles are located just as for type A. A center is located only at the origin. In region G_1, filled with closed trajectories, $0 \leqslant h < h_c$.

C) $\alpha \leqslant -\beta$ (Fig. 1c). All trajectories are closed and encircle the origin, and on them $h \geqslant 0$.

3. **Condition for the generation of a limit cycle.** The limit cycles of a perturbed system (1.2) with small σ and γ must be sought close to those trajectories of the unperturbed system (2.1), along which the integral of the perturbation equals zero, i.e.,

$$\oint_L 2\rho(\sigma - 2\gamma\rho)\, d\varphi = 0 \tag{3.1}$$

where L is a closed trajectory of (2.1) and the function $\rho(\varphi)$ is taken along L (see Chapter XIII in [3]). Therefore, to study the disposition of the limit cycles we need to seek trajectories L satisfying condition (3.1), viz., trajectories from which limit cycles are generated. Let L lie in G_m and let $H = h$ on L. By $G^m(h)$ we denote the region bounded by L. By passing in (3.1), with the aid of the Green's formula, from integration along L to integration over $G^m(h)$ and by introducing, following [1], the function $k_m(h)$, viz., the square of the radius of inertia of region $G^m(h)$, we rewrite (3.1) as

$$k_m(h) = w, \quad w = {}^1/_2\sigma/\gamma \tag{3.2}$$

$$k_m(h) = \frac{I_{2,m}(h)}{I_{1,m}(h)}, \quad I_{1,m}(h) = \int\limits_{G^m(h)} d\rho\, d\varphi$$

$$I_{2,m}(h) = \int\limits_{G^m(h)} 2\rho d\rho\, d\varphi, \quad m = 1, 2, 3$$

The behavior of the roots of these equations as a function of w is determined by the behavior of functions $k_m(h)$. The graphs of functions $k_m(h)$ $(m = 1, 2, 3)$ for various values of parameters α and β are shown in Fig. 2. The following statement describes the necessary properties of these functions.

Theorem 1.1*. Let the phase pattern of (2.1) be of type A $(\alpha > \beta)$. Then the nature of the behavior of functions $k_m(h)$ is determined by the value of α/β.

A1). If $\alpha/\beta > \xi_*$, where ξ_* is a constant defined below, $\xi_* \approx 4.11$, then

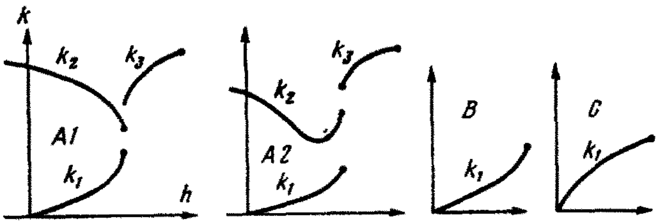

Fig. 2

1) in their domains $k_1(h)$ and $k_3(h)$ increase monotonically and $k_2(h)$ decreases monotonically as h grows; $k_m'(h) \neq 0$ $(m = 1, 2, 3,$ the prime denoted the derivative with respect to $h)$;

2) $k_1'(h) \to +\infty$ as $h \to h_c - 0$, $k_2'(h) \to -\infty$ as $h \to h_c - 0$, $k_3'(h) \to +\infty$ as $h \to h_c + 0$; $k_2(h) \to +\infty$ as $h \to -\infty$;

3) $k_1(h_c) < k_2(h_c) < k_3(h_c)$.

A2). If $1 < \alpha/\beta < \xi_*$, then

1) $k_2(h)$ has a nondegenerate minimum at some point h_d : $k_2'(h_d) = 0$, $k_2''(h_d) > 0$, $k_2'(h) \neq 0$ for $h \neq h_d$;

2) $k_2'(h) \to +\infty$ as $h \to h_c - 0$;

3) $k_2(h_d) > k_1(h_c)$;

4) in other respects the behavior of k_m is the same as in case A1.

A*). If $\alpha/\beta = \xi_*$, then $k_2'(h)$ tends to a finite negative limit as $h \to h_c - 0$; in other respects the behavior of k_m is the same as in case A1.

2°. Let the phase pattern of (2.1) be of type B $(\beta \geqslant \alpha > -\beta)$. Then k_1 behaves in the same way as in the pattern for type A.

3°. Let the phase pattern of (2.1) be of type C $(\alpha \leqslant -\beta)$. Then $k_1'(h) > 0$ and $k_1(h) \to +\infty$ as $h \to +\infty$.

In what follows we shall denote $w_m = k_m(h_c)$ $(m = 1, 2, 3)$, $w_a = k_3(h_a)$, $w_d = k_2(h_d)$. The cases $\alpha/\beta > \xi_*$, $\alpha/\beta = \xi_*$, $\xi_* > \alpha/\beta > 1$, $1 \geqslant \alpha/\beta > -1$, and $\alpha/\beta \leqslant -1$ will be called cases A1, A*, A2, B and C, respectively.

In Theorem 1, $\xi_* = (3 + \cos\vartheta_*) / (1 - \cos\vartheta_*)$, where ϑ_* is the (obviously single) root of the equation $\operatorname{tg}\vartheta - \vartheta = \pi$ examined for $\vartheta \in (0, \pi/2)$; $\vartheta_* \approx 1.352$, $\xi_* \approx 4.11$.

Theorem 1 is proved in Sects. 6.1 –6.4.

4. Bifurcations of the limit cycles of the perturbed system. Theorem 1 enables us to describe the bifurcations of the roots of the equation $k_m(h) = w$ and, hence, the bifurcations of the trajectories of the unperturbed problem (2.1), generating the limit cycles of (1.2). The bifurcations are seen from the graphs of $k_m(h)$ (Fig. 2). The following statement describes the bifurcations of the limit cycles of (1.2).

Theorem 2. For specified α, β, τ, satisfying the inequalities $\tau \neq 0$, $\beta \neq 0$, $\beta \neq |\alpha|$, $\alpha / \beta \neq \xi_*$, we can find $\delta > 0$ such that when $0 < |\sigma| + |\gamma| < \delta$

1) the bifurcations of the limit cycles of (1.2) are the same as the bifurcations generating the cycles of the trajectories of (2.1) for the same α, β, τ;

2) the collection of values of parameter w, at which the bifurcations take place in case A2, viz., $\{0, w_1', w_d', w_2', w_3', w_a'\}$, satisfy the bounds $|w_s - w_s'| < c_1 (|\sigma| + |\gamma|)$ $(s = 1, 2, 3, a, d)$, where $c_1 > 0$ is independent of σ and γ. Similar assertions are valid for cases A1, B and C;

3) when $w \neq w_d'$ all cycles are nondegenerate.

In particular, in case A2 the bifurcations as w grows take place in the following manner. Limit cycles do not exist when $w < 0$. When $w = 0$ a limit cycle is generated at the origin. It expands as w grows and when $w = w_1'$ it turns into separatrices, i.e., connects saddle singular points. Limit cycles do not exist when $w_1' < w < w_d'$. A double limit cycle is generated in region G_2 when $w = w_d'$. Under a further growth of w its constituent cycles diverge. When $w = w_2'$ one of them turns into separatrices and then disappears. The other cycle exists for all $w > w_d'$ and goes off to infinity as w increases. Four symmetric cycles (loops of separatrices) are generated when $w = w_3'$. As w grows these cycles move away from the separatrices, become smaller, and when $w = w_d'$ they disappear into four foci located in region G_3. Since the cycles are nondegenerate when $w \neq w_d'$ a change of stability of this point takes place as the cycle branches off from the singular point. The bifurcations of the limit cycles are described analogously in case A1, B and C. They were predicted in [1]. The proof of Theorem 2 is based on Theorem 1 and is carried out in the usual manner by using the bounds for the successor function [3, 4]. It is rather cubersome and is not given here.

5. Phase pattern of the perturbed system. If the perturbation in (1.2) is sufficiently small $(0 < |\sigma| + |\gamma| \ll 1)$, then the phase pattern, as compared with the pattern of the unperturbed system (2.1), changes in the following way. The saddles are displaced. The centers are displaced and turn into foci. The separatrices, in general, split up and cease to join singular points. Limit cycles emerge. As $t \to \pm\infty$ the remaining trajectories wind onto the foci and the limit cycles or go off to infinity. The type of the phase pattern is completely determined by the disposition and the nature of the singular points, separatrices, and limit cycles. The disposition of the limit cycles has been described in Sect. 4. When $w < -1$ the foci are stable if $\gamma > 0$ and unstable if $\gamma < 0$. This can be established by computing the eigenvalues in the first approximation with respect to σ and γ. The stability or instability of the foci and the limit cycles for the other values of w is determined by considering the birfurcations.

Thus, we can describe all the phase pattern types possible for the specified α, β, τ and for sufficiently small σ and γ. The phase pattern for case A2 is shown in Fig. 3a–c for $w = w_d'$, $w_d' < w < w_2'$, $w = w_2'$ $(\gamma > 0)$. The other possible types of phase pattern were shown in Fig. 3 in [1]. Under

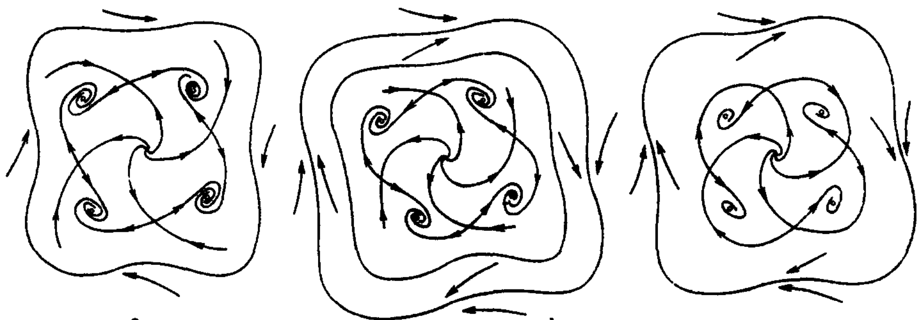

Fig. 3

a simultaneous change of signs of γ and σ stability changes to instability and vice versa.

6. **Proof of Theorem 1.** We assume that the phase pattern of (2.1) is of type A $(\alpha > \beta > 0)$. We omit the theorem's proof for the case when the phase pattern of (2.1) is of type B or C, because it is a literal repetition of the arguments presented below in Sect. 6.2 when describing the behavior of $k_1(h)$ for the phase pattern of type A. In the computations for the trajectories from region G_m we drop the index m and write k, I_1, I_2 and $G(h)$ instead of k_m, $I_{1,m}$, $I_{2,m}$ and $G^m(h)$ (see formulas (3.2)).

6.1. **Behavior of $k(h)$ in region G_3.** In G_3 we consider

$$k' = R / I_1^2, \quad R = I_2'I_1 - I_1'I_2$$

where the prime denotes the derivative with respect to h. Let us show that $k' > 0$.

From the relation $H = \tau\rho - \rho^2(\alpha + \beta\sin 4\varphi) = h$ we have that

$$\rho = \rho_{1,2} = {}^1/_2\,(\tau \mp \sqrt{v}) / u \tag{6.1}$$
$$u = u(\varphi) = \alpha + \beta\sin 4\varphi, \quad v = v(\varphi) = \tau^2 - 4hu(\varphi)$$

on the phase trajectory. Then

$$I_1 = \int\limits_{G(h)} d\rho\, d\varphi = \int\limits_{\varphi_-}^{\varphi_+} (\rho_2 - \rho_1)\, d\varphi, \quad I_2 = \int\limits_{G(h)} 2\rho\, d\rho\, d\varphi =$$
$$\int\limits_{\varphi_-}^{\varphi_+} (\rho_2^2 - \rho_1^2)\, d\varphi$$

where $\varphi_\pm$ are the limits of the variation of φ on the trajectory. Hence

$$R = -2\tau \left[\int_{\varphi_-}^{\varphi_+} \frac{d\varphi}{u\sqrt{v}} \int_{\varphi_-}^{\varphi_+} \frac{\sqrt{v}d\varphi}{u} - \int_{\varphi_-}^{\varphi_+} \frac{d\varphi}{\sqrt{v}} \int_{\varphi_-}^{\varphi_+} \frac{\sqrt{v}d\varphi}{u^2} \right]$$

We pass from the two independent integrations over the segment $[\varphi_-, \varphi_+]$ to an integration over a square, by introducing two variables of integration φ_1 and φ_2 and symmetrizing the integrand with respect to them. Then

$$R = -\tau \int_{\varphi_-}^{\varphi_+} \int_{\varphi_-}^{\varphi_+} \left[\frac{\sqrt{v_1}}{u_1 u_2 \sqrt{v_2}} + \frac{\sqrt{v_2}}{u_2 u_1 \sqrt{v_1}} - \frac{\sqrt{v_1}}{u_1^2 \sqrt{v_2}} - \frac{\sqrt{v_2}}{u_2^2 \sqrt{v_1}} \right] d\varphi_1\, d\varphi_2$$

$$(u_i = u(\varphi_i), \quad v_i = v(\varphi_i))$$

We transform the integrand (we denote it r) in order to show that it is nonpositive. We obtain

$$r = \frac{(u_1 - u_2)(u_2 v_1 - u_1 v_2)}{u_1^2 u_2^2 \sqrt{v_1 v_2}} = -\frac{\tau^2 (u_1 - u_2)^2}{u_1^2 u_2^2 \sqrt{v_1 v_2}} \leqslant 0$$

Then $R > 0$ and $k' > 0$ for $h_c < h < h_a$, as asserted. From this same expression for R it follows that $k'(h) \to +\infty$ as $h \to h_c + 0$. Computations show that $k'(h_a) = (12\tau)^{-1}(1 + \alpha/\beta) \neq 0$.

6.2. Behavior of $k(h)$ in region G_1. Instead of studying $k(h)$ directly, in G_1 we consider the equation $k(h) = w$ which we rewrite (using (3.2)) as

$$I(h, w) = wI_1(h) - I_2(h) = 0 \tag{6.2}$$

$$I_1(h) = \int_0^{2\pi} \rho(h, \varphi)\, d\varphi, \quad I_2(h) = \int_0^{2\pi} \rho^2(h, \varphi)\, d\varphi \tag{6.3}$$

where the function $\rho(h, \varphi) = \rho_1(h, \varphi)$ has been defined by formula (6.1). We examine the behavior of $I(h, w)$ as a function of h for various w.

Lemma 1. If $I'(h, w) = 0$ at some point $h \in (0, h_c)$, then $I''(h, w) < 0$ at this point.

This lemma is proved below.

Corollary 1. For each w there exists no more than one point $h \in (0, h_c)$ at which $I'(h, w) = 0$. If such a point exists, function I has a nondegenerate maximum at it.

Indeed, if two such points were to exist, then by the lemma function I would have maxima at them. Then between them a minimum point would exist, which is forbidden by the lemma.

Corollary 2. If $I(h, w)$ has a maximum, then as w changes this maximum is continuously displaced and can vanish only at the endpoints of the interval $(0, h_c)$.

Let us now consider the behavior of I at the endpoints of $(0, h_c)$. Obviously, $I(0, w) = 0$ for any w and $I(h_c, w_1) = 0$, where $w_1 = I_2(h_c) / I_1(h_c)$. Further,

$$I'(h, w) = \int_0^{2\pi} (w - 2\rho) \frac{\partial \rho}{\partial h} \, d\varphi$$

It can be shown that $I'(0, w) = 2\pi w / \tau$. At the saddle points (when $h = h_c$ and $\varphi = \pi / 8 + \pi n / 2$ $(n = 1, \ldots, 4)$) $\partial \rho / \partial h$ has singularities. It can be verified that $I'(h, w) \to -\infty$ as $h \to h_c - 0$, if $w < w_2 = 2\rho_c$, and $I'(h, w) \to +\infty$ as $h \to h_c - 0$, if $w > w_2$. A finite derivative $I'(h_c, w) > 0$ exists when $w = w_2$.

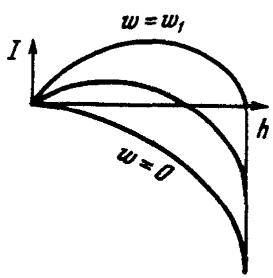

Fig. 4

The graph of $I(h, w)$ for $w = 0$ is shown in Fig. 4. As w grows the graphs corresponding to $h \neq 0$ rise upward. The corollaries to Lemma 1 and the information on the behavior of I at the endpoints of $(0, h_c)$ enable us to describe the evolution of the graph as w varies. Obviously, for $0 < w < w_1$ and $w = w_1$ the function $I(h, w)$ behaves as shown in Fig. 4 by the middle and upper curves, respectively. Hence, in particular, it follows that $w_2 > w_1$. From Fig. 4 it follows further that for $0 < w < w_1$ Eq. (6.2) has a single root on $(0, h_c)$ and that $I'(h, w) < 0$ at this root. Equation (6.2) has no roots on $(0, h_c)$ when $w \leqslant 0$ and $w \geqslant w_1$.

Now for any $h \in (0, h_c)$ we introduce $w = k(h) = I_2(h) / I_1(h)$. Then $I(h, w) = 0$ and by the preceding

$$k'(h) = -(wI_1' - I_2') / I_1 = -I'(h, w) / I_1 > 0$$

As $h \to h_c - 0$ we have $w \to w_1$ and $k'(h) \to +\infty$. It can be directly verified that $k'(0) = 1 / \tau > 0$. Consequently, in G_1 the function $k(h)$ behaves as was described in Theorem 1.

Proof of Lemma 1. Let $I'(h, w) = wI_1' - I_2' = 0$. Then

$$I''(h, w) = wI''_1 - I_2'' = J / I_1', \quad J = I_2' I_1'' - I_1' I_2''$$

Since $I_1' > 0$, we need to prove that $J < 0$. From (6.1) and (6.3) we get that in region G_1

$$\frac{\partial\rho}{\partial h}=\frac{1}{\sqrt{v}},\quad \frac{\partial^2\rho}{\partial h^2}=\frac{2u}{(\sqrt{v})^3},\quad \frac{\partial\rho^2}{\partial h}=-\frac{1}{u}+\frac{\tau}{u\sqrt{v}},\quad \frac{\partial^2\rho^2}{\partial h^2}=\frac{2\tau}{(\sqrt{v})^3}$$

$$J=2\tau\left[\int_0^{2\pi}\frac{d\varphi}{u\sqrt{v}}\int_0^{2\pi}\frac{ud\varphi}{(\sqrt{v})^3}-\int_0^{2\pi}\frac{d\varphi}{\sqrt{v}}\int_0^{2\pi}\frac{d\varphi}{(\sqrt{v})^3}\right]-2\int_0^{2\pi}\frac{d\varphi}{u}\int_0^{2\pi}\frac{ud\varphi}{(\sqrt{v})^3}$$

As in Sect. 6.1, we introduce $u_i = u(\varphi_i)$, $v_i = v(\varphi_i)$ and we pass to an integration over a square. Using the relation $v_1u_2 - u_1v_2 = \tau^2(u_2 - u_1)$, we obtain

$$J=\int_0^{2\pi}\int_0^{2\pi}\frac{A(\varphi_1,\varphi_2)}{(\sqrt{v_1}\sqrt{v_2})^3}\,d\varphi_1 d\varphi_2 \tag{6.4}$$

$$A(\varphi_1,\varphi_2)=[\tau^2(u_2-u_1)^2-u_2^2(\sqrt{v_1})^3-u_1^2(\sqrt{v_2})^3]/(u_1u_2)$$

We transform the numerator of the integrand to

$$A = A_{12}+A_{21}-2\tau^3,\quad A_{ij}=(u_j/u_i)\,[\tau^3-(\sqrt{v_i})^3]$$

Further, we have

$$A_{12}=\frac{u_2}{u_1}(\tau^2-v_1)\frac{\tau^2+\tau\sqrt{v_1}+v_1}{\tau+\sqrt{v_1}}$$

and an analogous formula for A_{21}. Since $(\tau^2-v_1)u_2=(\tau^2-v_2)u_1$,

$$A=-\frac{\tau(\sqrt{v_1})^3+v_1v_2}{\tau+\sqrt{v_1}}-\frac{\tau(\sqrt{v_2})^3+v_1v_2}{\tau+\sqrt{v_2}}$$

Hence it follows that $J<0$, as was required.

6.3. Behavior of $k(h)$ in region G_2. Similarly to Sect. 6.2, in G_2 we consider the equation

$$I(h,w)=wI_1(h)-I_2(h)=0 \tag{6.5}$$

for $-\infty<h\leqslant h_c$.

Lemma 2. If $I'(h,w)=0$ at some point $h\in(-\infty,h_c)$, then $I''(h,w)<0$ at this point.

This lemma is proved below. As in Sect. 6.2, from it, it follows that for each w we can find no more than one point on interval $(-\infty,h_c)$ at which $I'(h,w)=0$; if such a point is found, then $I(h,w)$ has a nondegenerate maximum at it and as w varies this maximum is displaced and can vanish only at point h_c.

Let us consider the behavior of $I(h,w)$ at the endpoints of interval $(-\infty,h_c)$.

Obviously, $I(h, w) \to -\infty$ as $h \to -\infty$ and $I(h_c, w_1) = 0$, where $w_1 = I_2(h_c) / I_1(h_c)$. Similarly to Sect. 6.2 it can be shown that when $w = w_2 = 2\rho_c$ the direction of the vertical tangent at point h_c is changed: $I'(h, w) \to +\infty$ as $h \to h_c - 0$, if $w < w_2$, and $I'(h, w) \to -\infty$ as $h \to h_c - 0$, if $w > w_2$. A finite derivative $I'(h_c, w) > 0$ exists when $w_1 = w_2$.

The character of the evolution of the roots of Eq. (6.5) as w varies is determined by the relation between w_1 and w_2. If $w_1 < w_2$, then as w varies the graph of $I(h, w)$ changes as shown in Fig. 5 (the middle curve corresponds to $w_1 < w < w_2$). The points of the graph are raised upward as w grows. There are no roots when $w < w_1$. For all $w > w_1$ Eq. (6.5) has a single root on $(-\infty, h_c)$ and at it $I'(h, w) > 0$.

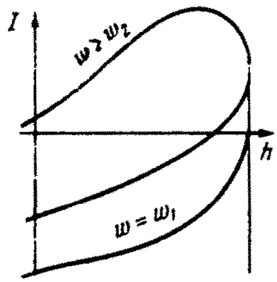

Fig. 5

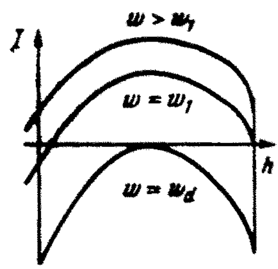

Fig. 6

If $w_1 > w_2$, then as w varies the graph of $I(h, w)$ changes as shown in Fig. 6. For w close to w_2, $w > w_2$, the function $I(h, w)$ has a single nondegenerate maximum and (6.5) has no roots. As w increases the points of the graph are raised upward. A double root at point $h = h_d$ appears for some $w = w_d$, $w_2 < w_d < w_1$. For $w_d < w < w_1$ there are two roots on $(-\infty, h_c)$, lying on different sides of point h_d; to the left of them $I'(h, w) > 0$ and to the right, $I'(h, w) < 0$. When $w = w_1$ the root on the right falls into point h_c; for $w > w_1$ a single root exists and $I'(h, w) > 0$ at it. If $w_1 = w_2$, then for $w > w_1$ a single root exists and $I'(h, w) > 0$ at it.

Now for any $h \in (-\infty, h_c)$ we introduce $w = k(h)$. Similarly to Sect. 6.2 we obtain

$$k'(h) = -I'(h, w) / I_1(h), \quad k''(h) = -(I''(h, w) + 2k'(h)I_1'(h)) / I_1(h)$$

If $w_1 < w_2$, then by the preceding the quantitiy $k'(h)$ is negative and $k'(h) \to -\infty$ as $h \to h_c - 0$. If $w_1 > w_2$, then $k'(h)$ is positive for $h > h_d$, is negative for $h < h_d$, and vanishes for $h = h_d$; however, $k''(h_d) > 0$; $k'(h) \to +\infty$ as $h \to h_c - 0$. If $w_1 = w_2$, then the quantity $k'(h)$ is negative and tends to a finite negative limit as $h \to h_c - 0$. In all cases, $k(h) \to +\infty$ as $h \to -\infty$. Thus, for $w_1 < w_2$, $w_1 > w_2$ and $w_1 = w_2$ the function $k(h)$ behaves as described in parts A1, A2 and A*, respectively, of Theorem 1.

L e m m a 3. The cases $w_1 < w_2$, $w_1 > w_2$ and $w_1 = w_2$ are realized if, respectively, $\alpha/\beta > \xi_*$, $\alpha/\beta < \xi_*$ and $\alpha/\beta = \xi_*$, where the quantity ξ_* has been introduced in Sect. 3.

Thus, the assertions of Theorem 1 concerning the behavior of $k(h)$ in region G_2 are valid. Lemma 3 is proved below.

P r o o f o f L e m m a 2. Since $I_1' < 0$, by analogy with Lemma 1 we need to prove that $J > 0$. The expression for ρ in region G_2 differs from the expre- ssion for ρ in region G_1 (see (6.1)) by the sign before the radical; therefore, the expression for J is obtained from (6.4)) by changing the signs before the radicals. Obviously, $J > 0$, as required.

P r o o f o f L e m m a 3. Using the definition of w_1 and w_2, we rewrite the relation $w_1 = w_2$ in the form

$$2\rho_c I_1(h_c) - I_2(h_c) = 0 \tag{6.6}$$

Here the functions $I_{1,2}(h_c)$ are determined by formulas (6.3) into which we need to substitute $\rho = \rho_2(h_c, \varphi)$. From (6.1) it follows that

$$\rho_{1,2}(h_c, \varphi) = \bar{\rho}_{1,2}(\varphi) = \rho_c (1 \pm \eta \,|\sin(2\varphi - \pi/4)|)^{-1}, \quad \eta = \sqrt{2\beta/(\alpha+\beta)} \tag{6.7}$$

Then (6.6) can be rewritten as

$$\int_0^\pi \left(\frac{2}{1 - \eta \sin\psi} - \frac{1}{(1 - \eta \sin\psi)^2} \right) d\psi = 0 \tag{6.8}$$

The integral in (6.8) is taken with the aid of the substitution $\operatorname{tg}(\psi/2) = s$ and (6.8) is reduced to

$$\frac{1 - 2\eta^2}{(1 - \eta^2)\sqrt{1 - \eta^2}} \left[\operatorname{arctg} \sqrt{\frac{1-\eta}{1+\eta}} + \operatorname{arctg} \frac{\eta}{\sqrt{1 - \eta^2}} \right] - \frac{\eta}{2(1 - \eta^2)} = 0 \tag{6.9}$$

We introduce $\vartheta = 2\arcsin\eta$, $\vartheta \in (0, \pi)$. Then (6.9) is rewritten $\operatorname{tg}\vartheta - \vartheta = \pi$. If ϑ_* is a root of this equation and $\eta = \sin(\vartheta_*/2)$, then

$$\alpha/\beta = (2 - \eta^2)/\eta^2 = (3 + \cos\vartheta_*)/(1 - \cos\vartheta_*) = \xi_*$$

Consequently, $w_1 = w_2$ when $\alpha/\beta = \xi_*$. It can be verified that $w_1 < w_2$ and $w_1 > w_2$ when $\alpha/\beta > \xi_*$ and $\alpha/\beta < \xi_*$, respectively, as was asserted.

6.4. R e l a t i v e d i s p o s i t i o n o f t h e c h a r a c t e r i s t i c p o i n t s o n t h e g r a p h s. To complete the proof of Theorem 1 we need to show that $k_1(h_c) < k_2(h_c) < k_3(h_c)$, and additionally in case A2 that $k_1(h_c) < k_2(h_d)$. From formulas (6.7) it follows that the inequality $\bar\rho_1(\varphi) < \rho_c < \bar\rho_2(\psi)$ is fulfilled for all $\varphi \neq \pi/8 + \pi n/2$ and $\psi \neq \pi/8 + \pi n/2$ $(n = 1, \ldots, 4)$. Further,

$$k_m(h_c) = \frac{I_{2,m}(h_c)}{I_{1,m}(h_c)}, \quad I_{j,m}(h_c) = \int_0^{2\pi} \bar\rho_m{}^j \, d\varphi \quad (m = 1, 2;\ j = 1, 2)$$

Then

$$k_2(h_c) - k_1(h_c) = \Big[\int_0^{2\pi}\int_0^{2\pi} \bar\rho_1(\varphi)\bar\rho_2(\psi)(\bar\rho_2(\psi) - \bar\rho_1(\varphi))\, d\varphi\, d\psi\Big] \times [(I_{1,1}(h_c) I_{1,2}(h_c)]^{-1} > 0$$

The other inequalities are proved similarly.

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REFERENCES

1. A r n o l ' d, V. I., Loss of stability of self-oscillations close to resonance and versal deformations of equivalent vector fields. Funktsial'nyi Analiz i Ego Prilozheniia, Vol. 11, No. 2, 1977.

2. B o g d a n o v, R. I., Versal deformation of a singular point of a vector field on a plane in the case of zero eigenvalues. Proc. I. G. Petrovskii Seminar, No. 2. Moscow, Izd. Mosk. Gos. Univ., 1976.

3. A n d r o n o v, A. A., L e o n t o v i c h, E. A., G o r d o n, I. I. and M a i e r, A. G., Theory of Bifurcations of Dynamic Systems on a Plane. Moscow, "Nauka", 1967.

4. B o g d a n o v, R. I., Bifurcations of a limit cycle of one family of vector fields on a plane. Proc. I. G. Petrovskii Seminar, No. 2, Moscow, Izd. Mosk. Gos. Univ., 1976.

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